

The variable-order fractional Fourier transform: A new tool for efficient reconstruction of images encoded by linear and quadratic gradients with reduced sensitivity to calibration errors

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INTRODUCTION: Image reconstruction of non-Fourier data encoded using nonlinear field gradients has emerged as a topic of interest with the advent of techniques such as PatLoc [1], 4D-RIO [2], and O-Space imaging [3]. For projection methods, 4D-RIO and O-Space, the reconstruction has required the use of “brute force” iterative matrix solvers such as Conjugate Gradients (CG) [2] or Algebraic Reconstruction Technique (ART) [3], which involve explicit calculation of large matrices of order $N_{\text{samples}}N_{\text{coils}} \times N_{\text{voxels}}^2$. These solvers are time-consuming and highly susceptible to inconsistency in the matrix equations arising from off-resonance effects or gradient calibration errors. In this work we show, using O-Space encoding as an example, that simulated data encoded using second-order gradients can be reconstructed using a generalization of the Fourier transform called the fractional Fourier transform (FrFT). The FrFT holds the promise of providing faster and less memory-intensive reconstructions and greater robustness to calibration errors and off-resonance effects. The FrFT has been previously used to reconstruct conventional Cartesian phase-encoded images in the presence of field inhomogeneities that can be modeled as a quadratic function [4]. We generalize this treatment for axial-plane O-Space imaging, in which the X and Y gradients traverse a radial k-space trajectory while a quadratic $-(X^2+Y^2)$ rephasing gradient plays at constant amplitude during readout, translating the quadratic function to a different location in the FOV for each radial spoke.

THEORY: The fractional Fourier transform is a particular instance of a linear canonical transform which applies affine transformations in time-frequency space [5]. Whereas the conventional Fourier transform causes a 90-deg rotation in time-frequency space, the fractional Fourier transform causes rotation by an arbitrary angle, α . This rotation angle varies between $-\pi$ and π , with values outside this range repeating the behavior in this interval modulo 2π . The Fourier transform corresponds to $\alpha = \pm\pi/2$, while $\alpha = 0$ reduces to the identity operator. The FrFT integral operator, $FrFT_{\alpha}\{f(u)\}$, is defined in one dimension as:

$$FrFT_{\alpha}\{f(u)\} = f(\rho(t)) = \sqrt{1 - i \cot(\alpha)} \exp(i\pi \cot(\alpha) \rho^2) \int \exp(-i2\pi \csc(\alpha) \rho u) \exp(i\pi \cot(\alpha) u^2) f(u) du \quad (1)$$

where $f(u)$ is the object, u is a unitless normalized spatial coordinate, $u = x\sqrt{N}/FOV$, given N voxels in the FOV, and ρ is a normalized frequency-domain coordinate. To cast the acquired MR signal in this form, let $k(t)$ and $l(t)$ denote the accumulated gradient moments of the linear, $(\gamma/2\pi)G(t)$, and quadratic, $(\gamma/2\pi)H(t)$, fields, respectively: $k(t) = -\gamma \int_0^t G(r) dr$, $l(t) = -\gamma \int_0^t H(r) dr$. If we then use the following mapping between gradient trajectories and FrFT parameters [6], $\alpha(t) = \cot^{-1}(-2l(t))$ the signal equation at time point t_n begins to resemble the FrFT integral:

$$s(t_n) = \int \exp(i\pi(-2k(t_n)u - 2l(t_n)u^2)) f(u) du = \int \exp(i\pi(u^2 \cot(\alpha_n) - 2u\rho_n \csc(\alpha_n))) f(u) du \quad (3)$$

Off-resonance effects can be described by an additional factor of $\exp(-i2\pi\Delta B_0 t)$ outside the integral. Multiplying $s(t)$ by a chirp function and constant then “completes the square” of the FrFT integral kernel: $f_n(\rho_n) = \sqrt{1 - i \cot(\alpha_n)} \exp(i\pi\rho_n^2 \cot(\alpha_n)) s(t_n)$. However, since the quadratic gradient plays during readout, α varies on a

point-by-point basis, preventing reconstruction from being performed by a single FrFT of $f_{\alpha}(\rho(t_n))$. Instead, we adapt the approach used in [4] to reconstruct Cartesian images in the presence of quadratic field inhomogeneity. We exploit the fact that the FrFT is a unitary operator by applying the conjugate of the forward integral kernel at each point in $s(t_n)$. We then modify the formalism in [4] by adding a k-space density compensation function (DCF) to make it compatible with the radial trajectory traversed by the linear gradients in O-Space imaging:

$$\hat{f}(u) = \csc(\alpha_n) \sum_{n=1}^{N_s} DCF(\rho_n) s(t_n) \exp(-i\pi(u^2 \cot(\alpha_n) - 2u\rho_n \csc(\alpha_n))) \quad (4)$$

METHODS: The above formalism is generalized to two dimensions and then used to reconstruct simulated O-Space echo signals using a single-uniform RF coil and a numerical reference phantom. For simplicity additive noise is omitted and will be treated in future work. Reconstructions of size 256×256 are shown based on simulated radial and O-Space echo signals for the case of a single, uniform RF coil, 512 readout samples, and 256 readout “spokes”. Off-resonance effects and errors in the calibrated gradient field, $H(u)$, are compared for ART and FrFT reconstructions.

DISCUSSION: FrFT reconstructions show markedly reduced sensitivity to uncorrected off-resonance effects and errors in gradient calibration, each of which cause deleterious phase cancelation in ART [7] and CG reconstructions. FrFT reconstructions degrade more gracefully, showing geometric distortions at the periphery where the quadratic field is strongest. There is a radial banding artifact visible in the FrFT O-Space reconstructions which may be a new form of truncation artifact. Modification of the radial k-space DCF may also be required to make it compatible with the FrFT in the presence of the quadratic field. Preliminary comparisons show a 2-to-3-fold improvement in reconstruction times using the FrFT as compared with ART, an advantage that is expected to grow as more efficient implementations of the variable-order FrFT are developed.

FUTURE WORK: Recent work has shown that a generalization of the 2-D FrFT known as the non-separable linear canonical transform allows the inclusion of second-order cross terms in the integral kernel, broadening the scope of the FrFT beyond the treatment of pure quadratics [8]. We are now investigating use of this more general FrFT to reconstructed signals simulated using hyperbolic gradients ($2XY$ and X^2-Y^2) such as those in second-order PatLoc and 4-D RIO. Another interesting direction for future work is the extension to multi-coil image reconstruction. We are exploring use of the frequency shift rule for the FrFT, a direct generalization of the analogous rule for the Fourier transform, $f_{\alpha}(\rho - \zeta \sin(\alpha)) = e^{i\pi\zeta^2 \sin(\alpha) \cos(\alpha)} e^{i2\pi\zeta \rho (1 - \cos(\alpha))} f(u) = \eta_{\alpha, \zeta} e^{i2\pi\zeta \rho (1 - \cos(\alpha))} f(u)$. This property may permit autocalibrated O-Space reconstructions by using multi-coil data to fill in missing lines of spectral information, a natural extension of GRAPPA parallel imaging [9] into the fractional Fourier domain. In addition to providing autocalibration of coil profiles, this approach is expected to introduce additional time savings as compared with ART, which backprojects each coil’s data separately.

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